3/5/19 Week 6 Class 1

2.2
[Start of Exam 2 → 25%]

2.2 → Determinants by Row Reduction

Hint: How do you know how to calculate determinants.

Theorem: Let A be a square matrix. If A has a row or col of zeros, then $\det(A) = \emptyset$.

Proof:
(i) Start by using cofactor expansion along THE ROW/COL containing all zeros.
(ii) Therefore, the determinants will be zero multiplied by the determinants of sub matrices.
(iii) So, $\det(A) = \emptyset$.

What is a determinant?
→ A property of a system ≈ decides something like 23andme DNA decoder

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Theorem: Let A be a square matrix $\det(A) = \det(A^T)$

$$\begin{bmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & a_{33} \ddots a_{nn} \end{bmatrix} \xrightarrow{A^T} \begin{bmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & a_{33} \ddots a_{nn} \end{bmatrix}$$

$\det(A) = a_{11} a_{22} a_{33} \cdots |a_{nn}|$ $\quad$ $\det(A^T) = a_{11} a_{22} \cdots (a_{nn})$

$\det(A) = \det(A^T)$

Proof with Class:
1. When calculating A^T, we swap the rows with the cols of A
2. Calculate the $\det(A)$ using cofactor expansion along row i.
3. Calculate the determinant of A^T using cofactor expansion along column i.
4. Hence ② must equal ③... $\det(A) = \det(A^T)$

ROW REDUCTION

⊙ Why use row reduction?
- Motivation for using row reduction to calculate determinants:
  → REF will always be triangular. → $\det(I) = 1$
- (1) We want to use row reduction to increase the number of zeros in the matrix, which will make calculating the determinant easier. Sometimes bringing the matrix to (R)REF first and then using cofactor expansion is the quickest way to calculate a determinant.

→ Just to introduce Øs

⊙ Impact to determinants by row op: (abbreviated version, full version online)

(i) $kR_i + R_j$ // Constant k times Row(i) + Row(j) // $\det(\text{old}) = \det(\text{new})$
↑ — NO CHANGE —

(ii) $R_i \xleftrightarrow{\text{swap}} R_j$ // Swap one row with another // $\det(\text{old}) = (-1)\det(\text{new})$
↑ — (-1) Coefficient — all matrices, swap → (-1) all rows

(iii) $R_i \to kR_i$ || Row i becomes the row times k || $\det(\text{old}) = k\det(\text{new})$

→ just one row will cost

↑ gotta keep track of all the constants

↑

(Ex) $\begin{vmatrix} 2 & 8 & -4 & 6 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 12 & -6 & 18 \end{vmatrix} \xrightarrow{R_1 \to 2R_1} 2\begin{vmatrix} 1 & 4 & -2 & 3 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 12 & -6 & 18 \end{vmatrix} \xrightarrow{-3R_1 + R_4 \to R_4} 2\begin{vmatrix} 1 & 4 & -2 & 3 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 9 \end{vmatrix}$

- This is not a matrix
- This is a determinant

$2\begin{vmatrix} 1 & 4 & -2 & 3 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 9 \end{vmatrix} \to (2)(9)\begin{vmatrix} 1 & 4 & -2 \\ 0 & 2 & -1 \\ 0 & 2 & 0 \end{vmatrix} = 2(9)(1)\begin{vmatrix} 2 & -1 \\ 2 & 0 \end{vmatrix} =$

$2(9)(1)(2) = \boxed{36}$

Most of the times you're going to use cofactors & reduction.

3/7/19 Week 6 Class 2

HW Section 2.2 #19

Evaluate the given determinants: $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$

⊙ What are we being asked to determine?
→ Find the determinant

⊙ Which one?
That one

$$\begin{vmatrix} a+g & b+h & c+c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$\begin{vmatrix} a+g & b+h & c+c \\ d & e & f \\ g & h & i \end{vmatrix} \xrightarrow{-R_3 + R_1 \to R_1} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$$

[#2.2 → #13, 2.2 → #22] · Gotta do homeworks, nothing is free.

P F S S M T (Exam)

1.3 | 1.8 | 1.2 | 1 (Proofs, P1, P2, Exam)

Add linearity condition (1.8) to exam guide → What two properties will it satisfy?

Ex1 → 6 to 7 Questions

(7)	2.1 :
(6)	1.8 :
(5)	1.7 :
(4)	1.6 :
(3)	1.5 :
(2)	1.4 :

Linear transformations: (be comfortable with this)

↳ We spoke about the transformation $T_A: \mathbb{R}^n \to \mathbb{R}^m$ where A is an $m \times n$ matrix.

↳ We spoke about T_A being linear if it satisfies the linearity conditions

1) $T(\hat{x}_1 + \hat{x}_2) = T(\hat{x}_1) + T(\hat{x}_2)$ → additive property

2) $T(c\hat{x}) = cT(\hat{x})$ → homogeneity property (scalar multiplication)

Ex: Evaluate: → plug in the solution vector
$T_A(\hat{x}_1 + \hat{x}_2) = T_A(\hat{x}_1) + T_A(\hat{x}_2)$

2.2: Determinants by Row Reduction

Q: Tell me about elementary row matrices → What does that mean in terms of determinants.

Q: What can you say about every elementary matrices?
- What can it BE?
- What can it NOT BE?
- It can't be $\emptyset$

- The determinant of an elementary matrix cannot be $\emptyset$ because it is one row operation away from I (1:10:00) → confusing

All elementary matrices are INVERTIBLE.

Advice: Translate definitions & Theorems into your own words.

(Back to 2.2):
Note:
→ The determinant of an elementary matrix WILL NEVER BE ZERO!

- Theorem: If A is a square matrix with two proportional rows (or columns) then $\det(A) = \emptyset$.

① Since A has two proportional rows, perform the row op $kR_i + R_j \to R_j$ when R_i and R_j are proportional. This will result in a row of zeros. Any matrix with a row of zeros has a determinant of zero.

Ex: R_i: $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{4}$ 1

R_j: 1 1 $\frac{1}{2}$ 2

2.3 Properties of Determinants & Cramer's Rule

Linearity condition $T(\hat{x}_1 + \hat{x}_2) = T(\hat{x}_1) + T(\hat{x}_2)$ additive property

Not generally true → $\det(A \pm B) \neq \det(A) \pm \det(B)$ // determinant operator is NOT Linear.
→ Also doesn't satisfy the homogeneity property

Statement: If A is an $n \times n$ Matrix, and k is any scalar, then the $\det(kA) = k^n \det(A)$.

Q: What is the significance of this statement?

↳ Homogeneity property → $T(c\hat{x}) = cT(\hat{x})$ ↳ determinants is not linear.

↳ Work on this proof on your own → HW

↳ What's a lemma? Like a theorem, used to prove another bigger theorem.

Lemma : (a side theorem)

↳ Axiom → something that is understood to be self evident, no proof necessary

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Lemma: If B is an $n \times n$ matrix and E is an elementary matrix, then $\det(EB) = \det(E)\det(B)$

//Proof by cases; going to show 1, same for other 2 ———

↳ Idea for proof: We will prove this for only one type of elementary row operation.

(i) If E is the elementary matrix created by multiplying R_i of the I_n matrix then the following product of EB can be calculated simply by multiplying R_i in B by k (a constant).

(ii) $E = \begin{bmatrix} 1 & & \emptyset \\ & k & \\ \emptyset & & 1 \end{bmatrix}_{n \times n}$, $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & \ddots & \\ b_{n1} & & & b_{nn} \end{bmatrix}$ $\xrightarrow{EB}$ ·diagonal matrix

$\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ kb_{21} & kb_{22} & \cdots & kb_{2n} \\ b_{31} & b_{32} & \cdots & b_{3n} \\ \vdots & & \ddots & \vdots \\ b_{n1} & & & b_{nn} \end{bmatrix}$ //(1)(R_1) //(k)(R_2)

$= k\det(B)$

(Left hand side proof)

- When the diagonal matrix is on the left quick shortcut is to multiply the diagonal to the rows of B

RHS: $\det(E)\det(B) = k\det(B)$ → (=) ← Proof Complete

Generalization of the above: - We can generalize this Lemma as $\det(E_1 E_2 \cdots E_r B) = \det(E_1)\det(E_2)\cdots\det(B)$.

Theorem: A square matrix A is invertible iff $\det(A) \neq \emptyset$.

Proof: (i) Let R be the RREF of matrix A

(ii) We can express R as $E_r E_{r-1} E_{r-2} \cdots E_1 A$, because each elementary row operation performed on A to get to R can be represented as multiplication on the left by an elementary Matrix.

$E_r E_{r-1} E_{r-2} \cdots E_1 A$ means we are performing elementary row operations on R

(iii) $\det(R) = \det(E_r)\det(E_{r-1})\cdots\det(E_1)\det(A)$ by Lemma

Q: Is there a relationship between $\det(R)$ and $\det(A)$

- if this is $\emptyset$ then ← this better not be $\emptyset$
- if this is 1 then ← this is 1

(iv) Notice, since the det of any elementary matrix is not Zero, the $\det(R)$ and $\det(A)$ are both either zero OR not zero.

(v) Given A is invertible, the RREF of A is I_n, by the equivalence. So R is I_n. So the $\det(I_n) = \det(R) = 1 \neq \emptyset$. Hence, $\det(A) \neq \emptyset$.

(This was one way gotta show the other way)

(vi) Given $\det(A) \neq \emptyset$, we know $\det(R) \neq \emptyset$. We know that the determinant of a square matrix in RREF that is NOT EQUAL to $\emptyset$ MUST have the identity as its RREF. This means $R = I_n$ by the equivalence theorem, A IS INVERTIBLE

*** Theorem: A square matrix A is invertible iff $\det(A) \neq \emptyset$. ***

Q: Why do we care if a matrix is or isn't Invertible?

What information do we get if a matrix is invertible?

(i) Quick way to solve a linear system $A\hat{x} = b \dots \hat{x} = A^{-1}\hat{b}$

(ii) We know that the determinant of the matrix $\neq \emptyset$.

(iii) The RREF of the matrix is the Identity

Theorem: If A and B are square matrices of the same size, then $\det(AB) = \det(A)\det(B)$.

(• Proof is proof by cases; 2 cases)

Case I If A is not invertible:

$\det(AB)$ ← (LHS): Since A is not invertible, the product AB is also NOT INVERTIBLE. This means $\det(AB) = \emptyset$.

$\det(A)\det(B)$ (RHS): Since A is not invertible, $\det(A) = \emptyset$ and $\det(B) = \emptyset$.

// The output of a determinant is $\mathbb{R}$ number

Case II If A is invertible:

(LHS): • Since A is invertible, we can express A as the product of elementary matrices by Equivalence theorem

$A = E_1E_2E_3\dots E_R$

• $AB = E_1E_2\dots E_RB \rightarrow \det(AB) = \det(E_1E_2\dots E_RB) = \det(E_1)\det(E_2)\dots\det(B)$

(RHS): • Since $A = E_1E_2\dots E_r \rightarrow \det(A) = \det(E_1)\det(E_2)\dots\det(E_r)$

↳ $\det(A)\det(B) = [\det(E_1)\det(E_2)\cdots\det(E_r)\det(E_r)\det(B)]$

3/14/19 Week 7 Class 2

- 2.3 Continued
- 2nd Exam before Spring Break: 4/18
- This Monday, free hour exam review
- Lowest grade 15%

Theorem: If A is invertible, then the $\det(A^{-1}) = \frac{1}{\det(A)}$

Quick Proof

Proof: ① Since A is invertible, A^{-1} exists and, $\det(AA^{-1}) = \det(A)\det(A^{-1}) = \det(I) = 1$

② $\det(A)\det(A^{-1}) = 1 \rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$ // Determinant is invertible, so it can't be // $\emptyset$.

Definition: If A is an $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix $\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$ is called the Matrix of cofactors of A. (Cofactor Matrix)

// Cofactor Matrix

The transpose of the cofactor matrix is called the adjoint of A and is denoted $\text{adj}(A)$.

// We need the $\text{adj}(A)$ to prove 2 Theorems in ch. 2

* Interesting fact about cofactors: – As it turns out, if we multiply the entries (a_{ij}) of any row (or col) of A by any other row (or col) from the cofactor matrix and then sum up the products, the sum will equal zero.

Ex not invertible

$$A \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \rightarrow C = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

$R_{1_A} \cdot R_{2_C} = 6 - 24 + 18 = \emptyset$ ✓

$R_{2_A} \cdot R_{3_C} = -12 + 30 - 18 = \emptyset$ ✓

Theorem: If A is an invertible Matrix, then the $A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$

// 2nd Way of Calculating the Inverse of a Matrix

$$\boxed{A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)}$$

Proof: ① Multiply both sides of the equation by $A\det(A)$ (is a constant, position no objection)

$\Rightarrow \det(A) AA^{-1} = \frac{\det(A) \cdot A \cdot \text{adj}(A)}{\det(A)}$

($AA^{-1} = I$)

$$\boxed{\det(A) I = A \, \text{adj}(A)}$$

② Right Hand Side: A $\quad$ $\text{adj}(A) = C^T$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{21} & & \vdots \\ \vdots & & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

(Matrix Multiplication)

// Determinant of A $\quad$ $\begin{bmatrix} a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} \\ \emptyset \quad a_{21}C_{21} + a_{22}C_{22} + \cdots + a_{2n}C_{2n} \\ \\ \end{bmatrix}$ // determinant of A, not $\emptyset$ because A is invertible // Why? Refer to "interesting fact" above

$$= \begin{bmatrix} \det(A) & & \\ & \det(A) & \\ & & \ddots \\ & & & \det(A) \end{bmatrix}$$

$= \det(A) I \Rightarrow \frac{\det(A) I}{\det(A)} = \frac{A \, \text{adj}(A)}{\det(A)}$

$\Rightarrow A^{-1} I = \frac{A^{-1} A \, \text{adj}(A)}{\det(A)}$

$\Rightarrow A^{-1} = \frac{I \, \text{adj}(A)}{\det(A)}$

Provided the matrix is invertible $\Rightarrow \boxed{A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)}$

Theorem: Cramer's Rule • If $A\hat{x} = \hat{b}$ is a linear system of n equations in n unknowns such that $\det(A) \neq \emptyset$, then the system has a unique solution namely $\hat{x} = A^{-1}\hat{b}$.

The solution is specifically $x_i = \frac{\det(A_i)}{\det(A)}$ } Should be a number

$$x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, x_n = \frac{\det(A_n)}{\det(A)},$$

where A_j is the matrix obtained by replacing the j^{th} column of A with $\hat{b}$.

Proof for Cramer's Rule:

Substituting A^{-1} matrix

① Since the $\det(A) \neq \emptyset$, A^{-1} exists and $\hat{x} = A^{-1}\hat{b}$.

② By the previous theorem, the solution becomes $\hat{x} = \left[\frac{1}{\det(A)} \cdot adj(A)\right]\hat{b}$

$$\hat{x} = \frac{1}{\det(A)}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ C_{1n} & & & C_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} C_{11}b_1 + C_{21}b_2 + \ldots C_{n1}b_n \\ C_{12}b_1 + C_{22}b_2 + \ldots C_{n2}b_n \\ \vdots \\ C_{1n}b_1 + \ldots\ldots\ldots + C_{nn}b_n \end{bmatrix}$$

// $adj(A)$
$(n \times n) \cdot (n \times 1) = (n \times 1)$

$$x_1 = (C_{11}b_1 + C_{21}b_2 + \ldots C_{n1}b_n) \div \det(A)$$
$$x_2 = (C_{12}b_2 + \ldots\ldots\ldots C_{n2}b_n) \div \det(A)$$
$$\vdots$$
$$x_j = (C_{1j}b_1 + C_{2j}b_2 + \ldots\ldots + C_{nj}b_n) \div \det(A)$$

③ $A_j = \begin{bmatrix} a_{11} & a_{12} & \ldots & b_1 & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & b_2 & \ldots & a_{2n} \\ & & & \vdots & & \\ a_{n1} & \ldots & & b_n & \ldots & a_{nn} \end{bmatrix}$ (j^{th} column)

// Cofactor on the b-column

$\rightarrow \det(A_j) = (b_1C_{1j} + b_2C_{2j} + b_3C_{3j} + \ldots + b_nC_{nj})$

// Using cofactor expansion along the $\hat{b}$ column

④ Notice that the numerator for the j^{th} component in $\hat{x}$ from ② is $\det(A_j)$.

Q: Why would you want to use Cramer's Rule?
- The j^{th} component
- Maybe you only need the j^{th} unknown rather than all

Ch. 3

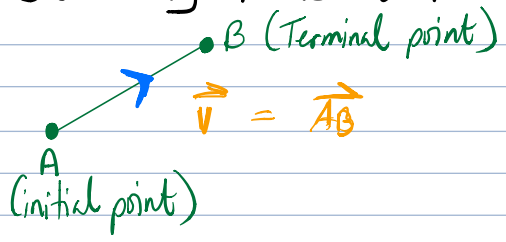
03:56:00

3.1: Vectors in 2-Space, 3-Space, and n-Space

- Q: What's a scalar? Scalars are just constants (ex. Speed, Distance)

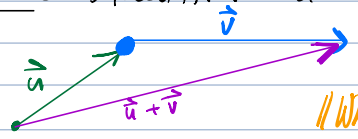
→ Vectors are scalars w/ a direction attached (ex. Velocity)

⊙ Geometrically vectors look like this:

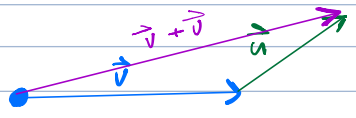


- Vectors of the same length and direction are equivalent.
- A vector whose initial point & terminal point are the same has a distance of $\emptyset$ and is called the zero vector. It is denoted $\vec{\emptyset}$.

Geometric Properties of Vectors ① Addition $\vec{u} + \vec{v}$ // Triangle rule



// What's $\vec{v}+\vec{u}$? Commute?



// Yes, $\vec{v}+\vec{u} = \vec{u}+\vec{v}$

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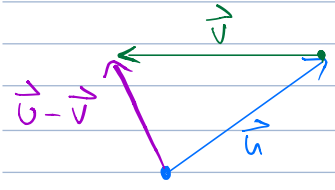
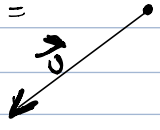
① Subtraction: $\vec{u} - \vec{v}$: same magnitude, opposite direction



① Negation: $-\vec{u}$ =



① Scalar multiplication – The scalar product of $\vec{v}$ by scalar c is a vector whose length is $|c|$ times the length of $\vec{v}$. The direction of $c\vec{v}$ is the same as $\vec{v}$ if $c > \emptyset$ and opposite if $c < \emptyset$. If $c = \emptyset$, then $c\vec{v} = \vec{0}$ and if $\vec{v} = \vec{0}$, $c\vec{v} = \vec{0}$.

Week 8, Class 1 3/19/19

Exam 1 → Average 70%. Highest 99%.

$E_1 \rightarrow 15\%$ $E_2 \rightarrow 25$ $\rightarrow 25$
$F \rightarrow 35\%$

3Ø Done from E_2

- Slater
- Math Lab → Berman → 231

Youtube → Patrick GMT
Ex: Strong

Do homework on time, then do homework again: Do HW Twice.

- 2.2 #23 – HW "I like this question a lot"

Use row reduction to show:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

① Motivation for row reduction?
① Triangular matrix

$= (b-a)(c-a)(c-b)$ $\xrightarrow[-a^2R_1+R_3\to R_3]{-aR_1+R_2\to R_2}$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & (b-a) & (c-a) \\ 0 & (b^2-a^2) & (c^2-a^2) \end{vmatrix}$$

⊙ What's the price to pay?

$$\begin{vmatrix} 1 & 1 & 1 \\ \emptyset & (b-a) & (c-a) \\ \emptyset & \emptyset & (c-a)(c-b) \end{vmatrix}$$

$-(b+a)R_2 + R_3 \to R_3$

Is this now a triangular matrix?

Side work:

$= -(b+a)(c-a) + c^2 - a^2$
$= -(b+a)(c-a) + (c-a)(c+a)$
$= (c-a)(-b-a+c+a)$
$= (c-a)(c-b)$

$= \boxed{(b-a)(c-a)(c-b)}$

3.1 Continued

→ Vectors that are scalar multiples of each other AND that have a common initial point are said to be collinear. AND parallel.

→ Vectors that are scalar multiples of each other BUT do not have a common initial point are parallel (not collinear)

→ Vectors in coordinate systems: recall the set of all real numbers can be viewed as the entire real number line and is denoted as $\mathbb{R}$.

→ The set of all ordered pairs of real numbers can be viewed geometrically as all possible points in the coordinate system (XY plane) and is denoted as $\mathbb{R}^2$.

→ The set of all ordered 3-tuples of real numbers can be viewed geometrically as 3D coordinates in space and is denoted as $\mathbb{R}^3$.

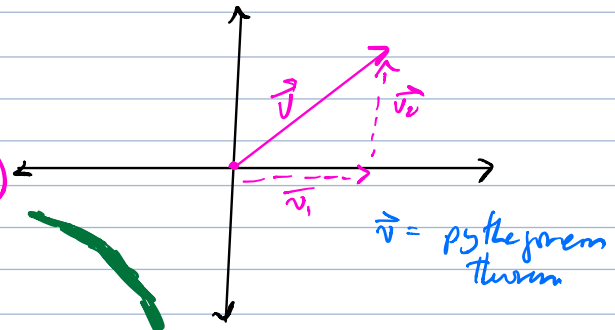
→ If a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ is positioned with initial point at the origin,

Side Question, what's the origin of $\mathbb{R}^2$? $(0,0)$
$\mathbb{R}^3$? $(0,0,0)$
$\mathbb{R}^n$? $(0,0,\dots 0_n)$

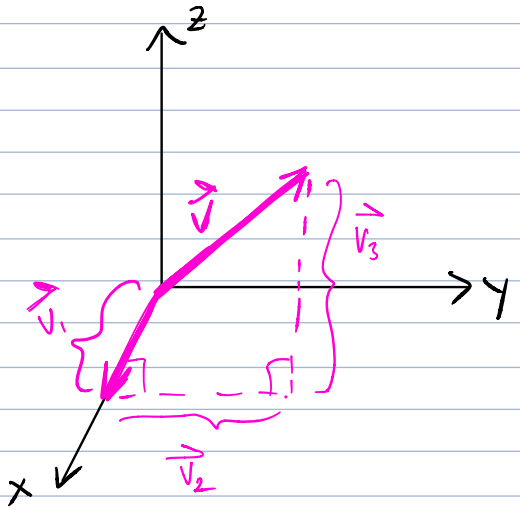
then the vector is determined by the coordinates of its terminal point. The vector coordinates are called vector components.

Ex Let $\vec{v}$ be in $\mathbb{R}^2$ so $\vec{v} = (v_1, v_2)$

Let $\vec{v}$ be in $\mathbb{R}^3$ so $\vec{v} = (v_1, v_2, v_3)$



Side Question: Can a higher dimension be described by lower dimensions?
~ "Degenerate box" → a rectangle?
~ Degenerate Dimension → set one of the dimensions to $\emptyset$.



* Some vectors will not have initial point at the origin

↳ If $\overrightarrow{P_1P_2}$ denotes the vector with initial point $P_1(x_1, x_2)$ and terminal point $P_2(x_2, y_2)$, the components of $\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$

Generalize to n-space

Definition: If n is a positive integer, then an ordered n-tuple is a sequence of n real numbers $(v_1, v_2, \dots v_n)$.
The set of all ordered n-tuples is called n-space and is denoted $\mathbb{R}^n$.

→ Operations in vectors in $\mathbb{R}^n$

Definition: Vectors $\vec{v} = (v_1, v_2, \dots v_n)$ and $\vec{w} = (w_1, w_2, w_3 \dots w_n)$ in $\mathbb{R}^n$ are said to be equivalent if $v_1 = w_1$, $v_2 = w_2$, $\dots$, $v_n = w_n$. We say $\vec{v} = \vec{w}$.

① Addition: $\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2, \dots, v_n + w_n)$ // Commutes

(ii) Subtraction: $\vec{v} - \vec{w} = (v_1 - w_1, w_2 - w_2, \dots v_n - w_n)$ // Does not commute

(iii) Scalar multiplication: $k\vec{v} = (kv_1, kv_2, kv_3, \dots, kv_n)$ for k some scalar

// These are all column/row vectors.

Helps with vector operations

Proofs not required

THEOREM 3.1.1 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *and* m *are scalars, then:*

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ → "additive inverse"

(e) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(f) $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

(g) $k(m\mathbf{u}) = (km)\mathbf{u}$ → "multiplicative commutation"

(h) $1\mathbf{u} = \mathbf{u}$ → effectively an identity

Reference

(i) $0\vec{v} = \vec{0}$ (j) $k\vec{0} = \vec{0}$

"Last thing on 3.1" In what context did we talk about linear combinations?

→ You can express them as standard basis vectors.

→ Linear Combinations:

Definition: If $\vec{w}$ is a vector in $\mathbb{R}^n$, then $\vec{w}$ is said to be a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ in $\mathbb{R}^n$ if it can be expressed in the form $\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + \dots + k_r\vec{v}_r$ where k_i's for $i = 1, \dots r$ are scalars. These scalars are called coefficients of the linear combination.

Ex Let $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ be vectors in $\mathbb{R}^n$, then $\vec{w}_1 = 2\vec{v}_1 - 13\vec{v}_2 + 7\vec{v}_3$ is a linear combination of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ in $\mathbb{R}^n$.

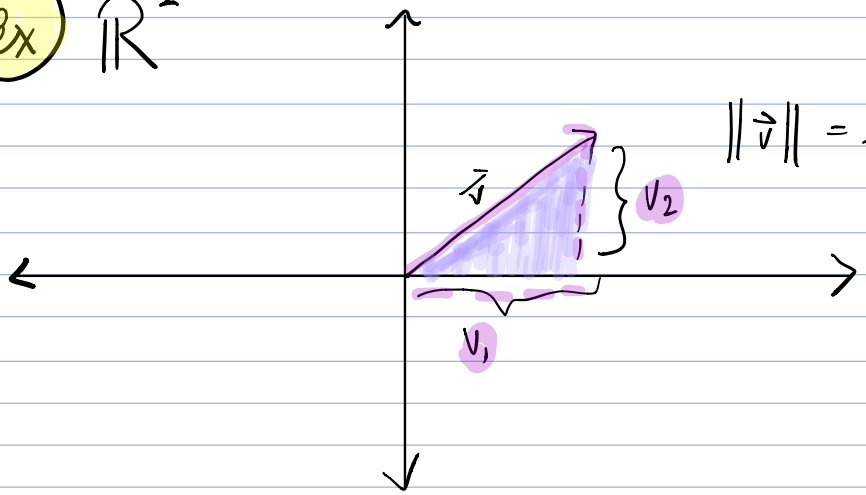
3.2

Norm (length), Dot Product, & Distance in $\mathbb{R}^n$

- The length of any vector $\vec{v}$ will be denoted as $\|v\|$.

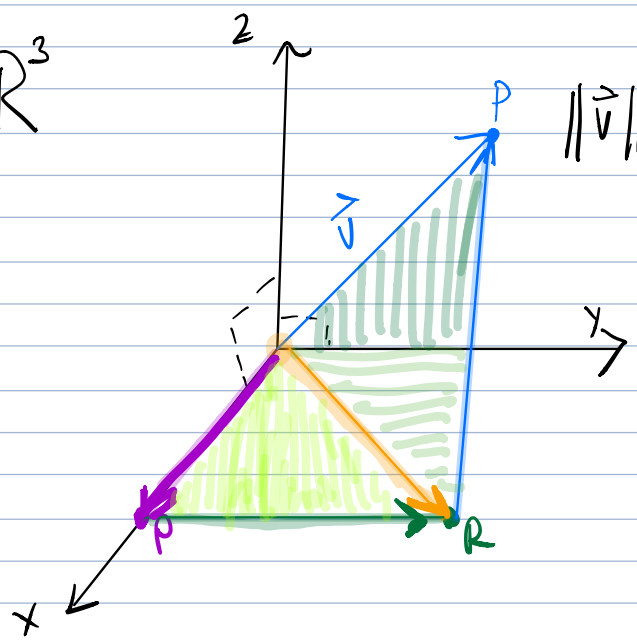
We will call this the norm, length, or magnitude, distance

Ex $\mathbb{R}^2$



$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$$

Ex $\mathbb{R}^3$



$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$\hookrightarrow = \sqrt{(\overline{OR})^2 + (\overline{RP})^2}$$

$$\hookrightarrow = \sqrt{(\overline{OQ})^2 + (\overline{QR})^2 + (\overline{RP})^2}$$

---

*We can generalize this process to $\mathbb{R}^n$:

Definition: If $\vec{v} = (v_1, v_2, \ldots, v_n)$ is a vector in $\mathbb{R}^n$ then the length is denoted by $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \ldots + v_n^2}$.

Week 8, Class 2, 3/21/19

- Prompts in problems.

Section 3.1 #17 → Let $\vec{u} = (1, -1, 3, 5)$, $\vec{v} = (2, 1, 0, -3)$.
Find scalars a and b so that $a\vec{u} + b\vec{v} = (1, -4, 9, 18)$

· distribute a & $\vec{u}$ $\quad a(1, -1, 3, 5) + b(2, 1, 0, -3) = (1, -4, 9, 18)$

$(a, -a, 3a, 5a) + 2b, b, 0, -3b = (1, -4, 9, 18)$

· create a matrix

a	$2b$	1
$-a$	b	-4
$3a$	0	9
$5a$	-3	18

· Solve the system of linear equations

$$\underset{4\times 2}{A}\ \underset{2\times 1}{\begin{bmatrix} a \\ b \end{bmatrix}} = \underset{4\times 1}{B}$$

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 3 & 0 \\ 5 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -4 \\ 9 \\ 18 \end{bmatrix}$$

→ Augmented Matrix: $[A|\vec{b}] = \left[\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & 1 & -4 \\ 3 & 0 & 9 \\ 5 & 3 & 18 \end{array}\right] \rightarrow \begin{array}{c} a\ \ b \\ \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$

$a = 3$
$b = -1$

· Calculator ~15s

· Now allowed to use calculator
Bring graphing Calculator.

∴ . . . → Finding on RREF using TI83Plus.

* Change dimensions of the matrix into square matrix.

Start playing around with Calculator.] Learn to convert decimal → fraction on calculator

---

3.2 (Continued) What is the length of a 0 vector? 0
What is the length of a unit vector? 1

Definition: A vector with length 1 is known as a unit vector. We use vectors when we are only concerned with direction (not length).

→ We can create a unit vector from any non-zero vector $\vec{v}$ in the direction of $\vec{v}$ as follows:

$$\vec{w} = \left(\frac{1}{\|\vec{v}\|}\right)\vec{v}$$

(scalar) (vector)

· This process of calculating a unit vector from a nonzero $\vec{v}$ is called normalization (or normalizing $\vec{v}$).

(Ex) Find the unit vector $\vec{u}$ with the same direction as $\vec{v}(-1,4,0,2)$

* Just gotta remember how to find the length of the vector via Pythagorean Theorem.

$\sqrt{(-1)^2+(4)^2+(0)^2+(2)^2} = \sqrt{1+16+4}$

$\sqrt{21}$ = ||length found ✓

$\sqrt{(-1)^2+(4)^2+(0)^2+(2)^2} = \sqrt{21}$ ← scalar

$\vec{u} = \frac{1}{\sqrt{21}}(-1,4,0,2)$

Answer: $\vec{u} = \frac{1}{\sqrt{21}}(-1,4,0,2)$] ||length is 1, because all unit vectors have length 1.

In vector spaces, we have standard unit vectors.

(Ex) $\mathbb{R}^2$, the standard unit vectors are $= \hat{i}(1,0)$ and $\hat{j}(0,1)$

$\mathbb{R}^3$, the standard unit vectors are $\hat{i}(1,0,0)\ \hat{j}(0,1,0), \hat{k}(0,0,1)$

$\mathbb{R}^n$, the standard unit vectors are $e_1(1,0,0\ldots,\overset{(n-1)}{0})$ ||Standard basis vector

$e_2(0,1,0\ldots\overset{(n-2)}{0})$

$\vdots$

$e_n(0,0,\ldots.1)$ ($n-1$ zeros)

* The Standard unit vectors are the standard basis vectors we studied in section 1.8.

↳ Notice, we can write any vector in $\mathbb{R}^n$ as a linear combination in terms of the standard unit vectors.

Q: How do you find the length between two points?

Distance in $\mathbb{R}^n$: if $u = (u_1, u_2, \ldots u_n)$ and $v = (v_1, v_2, v_3 \ldots v_n)$

(u: vector; $(u_1, u_2, \ldots u_n)$: coordinate)

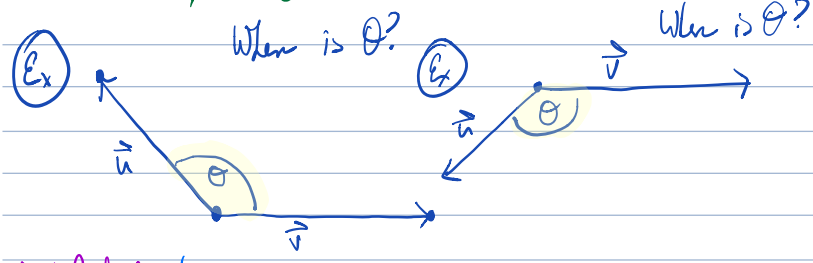
are points in $\mathbb{R}^n$, then we use the distance formula to calculate the length of the vector w/ initial point u and terminal v. We define the distance as $d(u,v) = \sqrt{(v_1-u_1)^2+(v_2-u_2)^2+\ldots+(v_n-u_n)^2}$

Multiplication of Vectors · dot product

① Vocabulary
② Example
③ Speak it in words

Let $\vec{u}$ and $\vec{v}$ be non-zero vectors in $\mathbb{R}^n$ with common initial point. The angle between $\vec{u}$ and $\vec{v}$ is the angle θ, where θ is to have only these values:

$0 \le \theta \le \pi$



Dot Product ① ②

Definition: If $\vec{u}$ and $\vec{v}$ are non-zero vectors in $\mathbb{R}^n$, and if θ is the angle between $\vec{u}$ and $\vec{v}$, then the dot product (also called the Euclidean Inner Product) of $\vec{u}$ and $\vec{v}$ is defined as $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$

(vectors → spits out a scalar)

→ every dot product has vector in left and right side.

* The dot product has inputs of vector and outputs a scalar.

Think of dot products as relationships between two vectors.

Ex Find the dot product of the vectors $\vec{u} = (0,0,1)$ and $\vec{v} = (0,2,2)$ w/ an angle $\theta = \frac{\pi}{4}$

(unit vector)

① length of $\vec{u}$

$\|u\| = \sqrt{0^2 + 0^2 + 1^2}$

$= \sqrt{1^2}$

$= 1$ ✓

① length of $\vec{v}$

$\|v\| = \sqrt{0^2 + 2^2 + 2^2}$

$= \sqrt{8}$

$= 2\sqrt{2}$ ✓

$= \|u\| \cdot \|v\| \cdot \cos\theta$

$= 1 \cdot 2\sqrt{2} \cos\left(\frac{\pi}{4}\right)$ (or $\frac{\sqrt{2}}{2}$)

$= 2\sqrt{2} \cdot \frac{1}{\sqrt{2}} =$

$= \frac{2\sqrt{2}}{\sqrt{2}}$

$= (2)$

$1 \cdot 2\sqrt{2} \cdot \cos\left(\frac{\pi}{4}\right)$

$= \frac{2\sqrt{2}}{1} \cdot \frac{\sqrt{2}}{2} = \sqrt{4} = (2)$

Dot Product Component Form

Definition: If $\vec{u} = (u_1, u_2, u_3 \ldots, u_n)$ and $\vec{v} = (v_1, v_2, v_3 \ldots, v_n)$ are vectors in $\mathbb{R}^n$, the dot product is defined by

$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$.

(whole bunch of scalars)

There is a good derivation in the textbook.

Ex Find dot product of $\vec{u}$ and $\vec{v}$ using component form. ///Must be in the same space.

$\vec{u} \cdot \vec{v} = (0)(0) + (0)(2) + (1)(2)$

$= 0 + 0 + 2$

$= 2$

Ex Does the following expression make mathematical sense?

$(\vec{u} \cdot \vec{v}) - \vec{w}$?

(Scalar) − (Vector) Doesn't make sense!

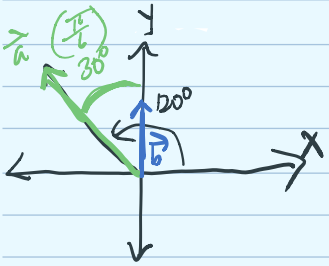
3/26 Week (9) Class 1

· final May 21st // 4 Classes before E

// What (Space) are you in? "X-Y plane".

HW Question (3.2 #13)

Given $\|\vec{a}\|$ is 9 $\|\vec{b}\|$ is 5



⊙ Find $\vec{a} \cdot \vec{b} = ?$

① Study Tips Reformat Questions

→ // Not always obvious.

// 2 formulas to choose from.

$\|\vec{a}\| \times \|\vec{b}\| \times \cos\theta_{(30°)}$ → // Know your trig

$= (9) \times (5) \times \cos(120°)$ → θ is direction independent

$= 45\cos 30° = 45\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{45\sqrt{3}}{2}}$

$\emptyset \le \theta \le \pi$

· default to positive

#7 (3.2) → Let v = (−2,3,0,6). Find all scalars k such that $\|k\,\mathbf{v}\| = 5$

→ $|k|\,\|\vec{v}\|$

① $\|\vec{v}\| = \sqrt{(-2)^2 + (3)^2 + (0)^2 + (6)^2} = \sqrt{49} = \pm 7$

$|k|\,\|\vec{v}\| = 7k = 5$

· scalar could be positive/negative

include ± to fulfill "all scalars" conditions.

$k = \frac{5}{7} \rightarrow \boxed{k = \pm\frac{5}{7}}$

Alternate Method: $k\vec{v} = (-2k, 3k, 0, 6k)$

$\sqrt{4k^2 + 9k^2 + 36k^2} = 5$

$4k^2 + 9k^2 + 36k^2 = 25$

$49k^2 = 25$

$\sqrt{k^2} = \sqrt{\frac{25}{49}}$

$\boxed{k = \pm\frac{5}{7}}$

⊙ This approach is better because you have to take the square root, thus including the ±

· What's another application of vectors?

3.2 : Algebraic Properties of the Dot Product

Unit vector dotted with itself is 1

(i) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ (ii) $\vec{u}\cdot(\vec{v}\pm\vec{w}) = \vec{u}\cdot\vec{v} \pm \vec{u}\cdot\vec{w}$

(iii) $(\vec{u}\pm\vec{v})\cdot\vec{w} = \vec{u}\vec{w} \pm \vec{v}\vec{w}$ (iv) $k(\vec{u}\cdot\vec{v}) = (k\vec{u})\cdot\vec{v} = \vec{u}\cdot(k\vec{v})$

(v) $\vec{v}\cdot\vec{v} = \emptyset$ iff, $\vec{v} = \vec{\emptyset}$

(vi) $\vec{0}\cdot\vec{v} = \vec{v}\cdot\vec{0} = \emptyset$ // scalar

// Proofs in book, get comfortable with HOW they work

(Ex) Simplify & Evaluate: $(3\vec{u} - \vec{v})\cdot(-2\vec{u} + 2\vec{v})$ // could you use foil?

$3\vec{u}\cdot -2\vec{u} - \vec{v}(-2\vec{u}) + -3\vec{u}2\vec{v} - \vec{v}2\vec{v} =$

$-6\vec{u}\vec{u} + 2\vec{v}\vec{u} + 6\vec{u}\vec{v} - 2\vec{v}\vec{v} =$

$-6\vec{u}\vec{u} + 8\vec{u}\vec{v} - 2\vec{v}\vec{v} =$ // Can a vector be squared?

$= -6\|\vec{u}\|^2 + 8\vec{u}\vec{v} - 2\|\vec{v}\|^2$

$\vec{u}\cdot\vec{u} = \|\vec{u}\|\,\|\vec{u}\|\cos(\emptyset)$

$= \|\vec{u}\|^2$

Nice Problem that catches all questions in one problem.

Q: What's the significance of dot product? You can use it to write equations for lines.

⇒ Tells you position, especially with $\emptyset$. When dot product is $\emptyset$, perpendicular lines.

(Orthogonal) ≈ Perpendicular Synonym.

(Jump to 3.5, 3.3, 3.4, finish Ch. 3) ~ Thinking of Exam on 3/11
(Ch. 2, 3, 4.1)

3.5

⊙ Cross Product

⊙ inputs vectors in $\mathbb{R}^3$
⊙ output $\vec{v}$ is $\mathbb{R}^3$

// only defined in $\mathbb{R}^3$, lives in $\mathbb{R}^3$

⊙ both $\vec{u}$ & $\vec{v}$ have same initial point

Definition: If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are vectors in $\mathbb{R}^3$ then the cross product $\vec{u} \times \vec{v}$ is the vector defined by

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - v_1u_2).$$

The cross product can be represented as $\vec{u} \times \vec{v} =$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

"False Determinant"

These are vectors

• Works the same way as a determinant

// Is this a determinant?
Not a determinant

By definition, the determinant is scalar.
• This has to be a vector in $\mathbb{R}^3$

⊙ This false determinant cofactor expansion only along the top row.

// What is the first component? $\hat{i}$

$$= \hat{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= (u_2v_3 - v_2u_3)\hat{i} - \hat{j}(u_1v_3 - v_1u_3) + \hat{k}(u_1v_2 - v_1u_2)$$

$$= (u_2v_3 - v_2u_3)\hat{i} + (v_1u_3 - u_1v_3)\hat{j} + (u_1v_2 - v_1u_2)\hat{k}$$

// Looks familiar?

• This is another way of writing a vector: $\hat{i}$ comp, $\hat{j}$ comp, $\hat{k}$ comp

Ex Find $\vec{u} \times \vec{v}$, where $\vec{u} = (2, 1, 5)$ and $\vec{v} = (3, 7, 6)$

Test Question!

$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - v_1u_2)$.

$(1)(6) - (5)(7),\ (5)(3) - (2)(6),\ (2)(7) - (3)(1)$

$6 - 35,\ 25 - 12,\ 14 - 3$

$= (-29,\ 13,\ 11)$ ✗

$\vec{u} \times \vec{v} =$

$\hat{i}$	$\hat{j}$	$\hat{k}$
2	-1	5
3	7	4

// or Expansion to Solve.

$= (-41, 3, 17)$ ✓

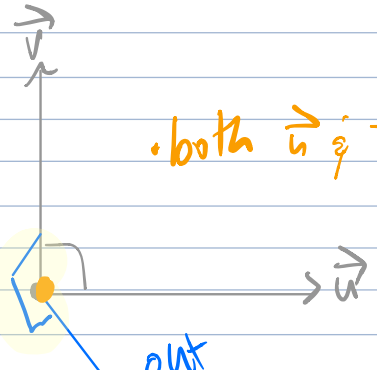
// Extremely Algorithmic

inputs → ? $\mathbb{R}^3$

outputs → $\mathbb{R}^3$

---

*Note: The cross-product vector is ORTHOGONAL to both input vectors.



• both $\vec{u}$ & $\vec{v}$ start at Origin, start at the same point.

↘ Cross product vector, orthogonal to part vectors.

Theorem • If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^3$, then (9:00:00)

ⓐ $\vec{u} \cdot (\vec{u} \times \vec{v}) = \phi$ and $\vec{v} \cdot (\vec{u} \times \vec{v}) = \phi$

ⓑ $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$

Try Proof

ⓑ $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 -$

"Scalar equals scalar"

↳ Proof

Step ①

LHS: $(u_2v_3 - u_3v_2)^2 + (u_3v_1 - v_3u_1)^2 + (u_1v_2 - v_1u_2)^2$

RHS: $(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) - (u_1v_1+u_2v_2+u_3v_3)^2$

Complete proof by setting each equal to the other

Step ② LHS = RHS // Factor

• The Right Hand Rule $\vec{u}\times\vec{v}$ ⇒ direction is out of board

Right hand thumb

(3/28/13) Week 9 Class 2 • 3 Classes Before E2

(3.5 continued) – Recall computing the cross product using coefficient expansion

Properties of the Cross Product — if $\vec{u}, \vec{v}, \vec{w}$ are vectors in $\mathbb{R}^3$ (cross product only in $\mathbb{R}^3$) and k is a scalar (can be $\emptyset$) then we get the following properties:

ⓐ $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$ (cross) (cross)

ⓑ $\vec{u}\times(\vec{v}\pm\vec{w}) = (\vec{u}\times\vec{v}) \pm (\vec{u}\times\vec{w})$

ⓒ $k(\vec{u}\times\vec{v}) = (k\vec{u})\times\vec{v} = \vec{u}\times(k\vec{v})$ // k to one or the other not both

ⓓ $\vec{u}\times\vec{\emptyset} = \vec{\emptyset}\times\vec{u} = \vec{\emptyset}$

ⓔ $\vec{w}\times\vec{w} = \vec{\emptyset}$

Cross Equalities

• What are the components of a $\vec{\emptyset}$? ⊙ (9:20:00)
↳ "Pho-Determinant"

Q: What are we doing with Cross Products? → Orthogonal vector to other vectors.
Q: How do we use this?

Recall: $\|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u}\cdot\vec{v})^2$

⊙ Looking at the RHS, $\|\vec{u}\|^2\|\vec{v}\|^2 - (\|\vec{u}\|\,\|\vec{v}\|\cos\theta)^2 =$

$= \|\vec{u}\|^2\|\vec{v}\|^2 - \|\vec{u}\|^2\|\vec{v}\|^2\cos^2\theta$

$= \|\vec{u}\|^2\|\vec{v}\|^2(1-\cos^2\theta)$

$= \|\vec{u}\|^2\|\vec{v}\|^2(\sin^2\theta)$

$$= \sqrt{\|\vec{u}\|^2 \|\vec{v}\|^2 (\sin^2\theta)}$$

$$\left[\ \|\vec{u}\times\vec{v}\| = \pm \|\vec{u}\|\|\vec{v}\| \sin\theta\ \right]$$

//End of 3.5

Theorem: Two nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel (p) iff $\vec{u}\times\vec{v} = \vec{\emptyset}$ (q)

Proof $p \to q$ (in class) try $q \to p$ (HW) //Possible Exam Q

① Since $\vec{u}$ and $\vec{v}$ are parallel, we know θ must be either $\emptyset$ or π radians

//We're trying to show q //Look at the magnitude of Cross Product

② $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$ Where in either case ($\theta = \emptyset$ or $\theta = \pi$)

$\sin(\theta) = \emptyset$ so $\|\vec{u}\times\vec{v}\| = \emptyset$

//Only 1 vector has a length of $\emptyset$.

③ Only the zero vector has length of $\emptyset$. Hence $\vec{u}\times\vec{v} = \vec{\emptyset}$

//Practice Proving $q \to p$

//What is the range of $\sin(\theta)$? $-1 \le f(x) \le 1$ //Review trig

---

3.3 more on dot products & orthogonality

3.3 – More on Dot Product & Orthogonality

Def: Two nonzero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are said to be orthogonal (perpendicular) if

$\vec{u}\cdot\vec{v} = \emptyset$.

//if the dot product of two vectors is $\emptyset$, the vectors are $\perp$ orthogonal.

//$\cos\left(\frac{\pi}{2}\right) = \emptyset$

Q// What is the dot product of $\vec{u}\cdot\vec{\emptyset} = ?$ $\emptyset$

$\vec{\emptyset}\cdot\vec{v} = ?$ $\emptyset$ //Symmetry

$\vec{\emptyset}\cdot\vec{\emptyset} = ?$ $\emptyset$

The $\vec{\emptyset}$ is Orthogonal to any and all vectors // Verify

By convention, the zero vector ($\vec{0}$) is orthogonal to all vectors in $\mathbb{R}^n$.

- Any vector crossed with itself → $\vec{0}$ is orthogonal to all vectors

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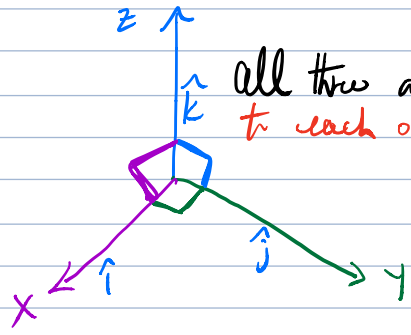
(Ex) Consider the standard basis vectors of $\mathbb{R}^3$ $(\hat{i}, \hat{j}, \hat{k})$



all three axes are Orthogonal to each other → Show that the 3 standard unit vectors in $\mathbb{R}^3$ are orthogonal to each other:

$\hat{j} \cdot \hat{k} = 0$

Review dot product $\|(0,1,0) \cdot (0,0,1)\| \cos\theta$

① $\hat{i} \cdot \hat{j}$ = use component form

$(1,0,0) \cdot (0,1,0) = (1)(0) + (0)(1) + (0)(0) = \emptyset$

② $\hat{j} \cdot \hat{k} = (0,1,0)(0,0,1) = (0)(0) + (1)(0) + (0)(1) = \emptyset$

③ $\hat{i} \cdot \hat{k} = \emptyset$

The cross product is constructed (scalar)(vectors) + (scalar)(vectors)...
Q: What are the vectors? → Linear Combinations

This makes sense because the computation fits. (10:10:00)

Yellow
Red: 255
Green: 255
Blue: $\emptyset$

Linear combination of Yellow: $255(R) + 255(G) + \emptyset(B)$

Purple: $230(R) + 20(G) + 216(B)$

→ Lines & Planes Determined by Normals & Points

- Equations of lines in $\mathbb{R}^2$: • slope-intercept
  • point-intercept

General equation of line → $Ax + By + C = \emptyset$

Recall, to write the equation of a line in $\mathbb{R}^2$, we need a point & a direction (slope).

To write a plane in $\mathbb{R}^3$, we need one point AND a direction (inclination)

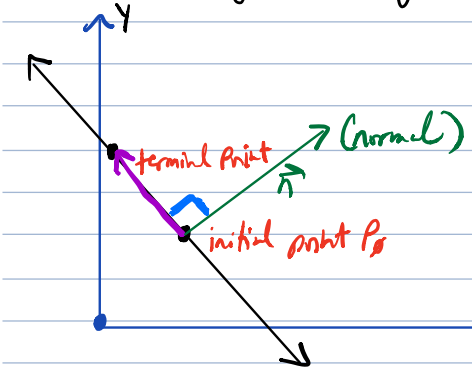
"Angle of inclination" denotes a direction (important) of the plane.

Similarly, we can write the equation of a plane in $\mathbb{R}^3$ with a point and a direction (inclination).
→ Key takeaway → you need point, direction

(Not in book) → We can use the dot product to come up with a direction. To do this, we will use a nonzero vector called the normal. Do not confuse this with the book's "normal"

↳ normal → a nonzero vector that is ORTHOGONAL to the line or plane.
We will denote normals as $\vec{n}$.

(Ex) The equation of a line in $\mathbb{R}^2$



$P(x, y)$
$P_0(x_0, y_0)$ where these are fixed constants
↳ Coordinates of a point (in $\mathbb{R}^2$)

$\vec{n} = (a, b)$ // components of a vector

$\overrightarrow{P_0P}$ = (terminal Point, initial Point)
$= (x - x_0, \ y - y_0)$

$\vec{n} \cdot \overrightarrow{P_0P} = 0$
$(a, b) \cdot (x - x_0, y - y_0) = 0$
→ Point-Normal form of the Eq. of a line

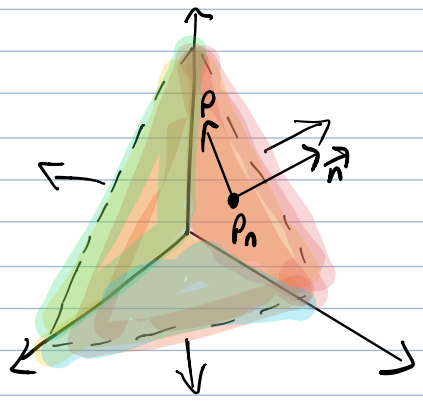
$a(x - x_0) + b(y - y_0) = 0$ // Equation of a line

General form of an Equation of a Line in $\mathbb{R}^2$: $\boxed{Ax + By + C = 0}$

$ax - ax_0 + by - by_0 = 0$
(ax_0: this is a constant; by_0: this is a constant)
↳ (C)

$ax + by + c = 0$

Ex Equation of a Plane in $\mathbb{R}^3$



4 sided pyramid

$P(x, y, z)$

$P_0(x_0, y_0, z_0)$

$\vec{n} = (a, b, c)$

$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$

$\boxed{\vec{n} \cdot \overrightarrow{P_0P} = \emptyset}$ → Point normal form

(Sub in coordinates)

$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = \emptyset$

$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = \emptyset}$

↳ General equation of a plane in $\mathbb{R}^3$

Prep for E2

W 6 → Ch.2 →
R 5 → Ch.3, Ch.2
F 4 → Ch.3, Ch.2
S 3 → Ch.3, Ch.2
S 2 → Ch.4, Ch.3, Ch.2
M 1 → Ch.4, Ch.3, Ch.2

(313)

(7)
323
355
370

320
331
343
340
(Wins)
241
E_1
E_2
(2)

Spanish 4
316 3
Numerical 3
Distributed System 3
E_3 3
E_4 3
(2)

(2020)
(graduation)

4/2/19 → Week 10, Class 1 (Exam next Thursday) 2.2, 2.3 Ch.3, (3.1–3.5) Calculator allowed

"Whose got HW Questions?"

3.5 TF (b), (f)

- What does coplanar mean? coincide, stacked on each side
- What is a normal vector to a plane? Orthogonal to the planes

(b) A normal vector to a plane

(f) 3 components "Fun Question"

⊙ Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^3$, $\vec{u}$ is a nonzero vector.

⊙ if $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, then $\vec{v} = \vec{w}$

⊙ Would $\vec{u} \times \vec{v}$ hold $\vec{u} \times \vec{w}$, then $\vec{v} = \vec{w}$

$\vec{u} = (1,1,1)$ $\vec{v} = (2,2,2)$ $\vec{w} = (0,0,0)$

• verify this is the case

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix} = \phi\hat{i} + \phi\hat{j} + \phi\hat{k} = \vec{\phi}$$

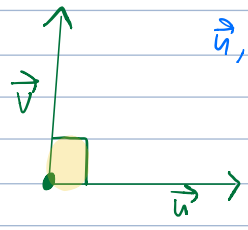
• Make one of these $\vec{0}$ vector

Collinear, scalar multiples

How do you know two vectors are collinear → scalar multiples of each other.

3.3 continued, finish 3.4 on Thursday

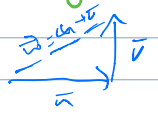
Projections: Insights →



$\vec{u}, \vec{v}$ are orthogonal

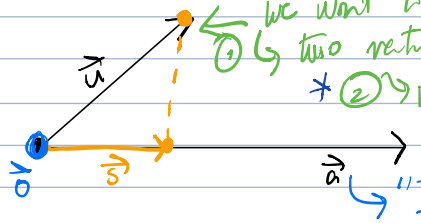
How do you add vectors?

↳ Triangle rule



Sometimes we will want to decompose a vector into two orthogonal components. We can do this using projections.

Example →



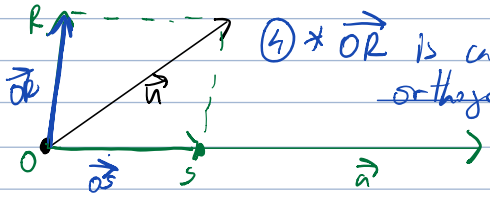
we want to decompose $\vec{u}$ into two orthogonal vectors.

① ↳ two vectors $\vec{0}, \vec{s}$, → $\vec{os}$

* ② ↳ vector $\vec{os}$ is called the vector projection of $\vec{u}$ "on to" $\vec{a}$. or "along"

↳ "I like to call $\vec{a}$, a reference vector"

③ "What if we subtract $\vec{os}$ from $\vec{u}$?" // switch direction and add



④ * $\vec{OR}$ is called the vector projection of $\vec{u}$ orthogonal to $\vec{a}$.

→ Derivation of Projection formulas

* if two vectors are collinear then they are scalar multiples.

① Notice $\vec{os} + \vec{OR} = \vec{u}$

② Also, notice, $\vec{os}$ is a scalar multiple of $\vec{a}$.
↳ So $\vec{os} = k\vec{a}$, where k is a nonzero scalar.

③ $\vec{u} = \vec{os} + \vec{or} = k\vec{a} + \vec{OR}$

// dot products are symmetric

④ $\vec{u} \cdot \vec{a} = (k\vec{A} + \vec{OR}) \cdot (\vec{a}) = k\vec{a} \cdot \vec{a} + \vec{OR} \cdot \vec{a} = k\|\vec{a}\|^2 + \phi = k\|a\|^2$

// vector dotted with itself is magnitude^2

↳ Rewrite → $\vec{u} \cdot \vec{a} = k\|a\|^2$ → solving for $k = \dfrac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}$

⑤ $\vec{OS} = k\vec{a}$ = ||direct substitute = $\left[\frac{(\vec{u}\cdot\vec{a})}{\|\vec{a}\|^2}(\vec{a})\right]$ → $\left(\frac{\text{Scalar}}{\text{Scalar}}\right)(\text{vector}) =$

= (Scalar)(vector) =

= scalar multiple

$\left[\text{proj}_{\vec{a}}\vec{u} = \frac{(\vec{u}\cdot\vec{a})}{\|\vec{a}\|^2}(\vec{a})\right]$

⑥ $\vec{OR} = \vec{u} - \vec{OS} = \left[\vec{u} - \frac{(\vec{u}\cdot\vec{a})}{\|\vec{a}\|^2}(\vec{a})\right] = \vec{u} - \text{proj}_{\vec{a}}\vec{u}$ → Make sure this makes mathematical sense.

Ex) ||Not in the book *Possible for Test*

ⓐ Find the vector components of $\vec{u}$ along $\vec{a}$ and orthogonal to $\vec{a}$ where $\vec{u} = (1,1,2)$ and $\vec{a} = (-2,3,1)$ (what?)

ⓑ Verify that $[\text{proj}_{\vec{a}}\vec{u}]$ is orthogonal to $[\vec{u} - \text{proj}_{\vec{a}}\vec{u}]$. How?

↳ •dot product •colinear •orthogonal •vector component ← none of these are solid in Reals.

① Find the [vector component] of $\vec{u}$ & $\vec{a}$ AND orthogonal to $\vec{u}$

$\vec{u} = (1,1,2)$ $\vec{a} = (-2,3,1)$ ||on three reference coordinates?

① $k = \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2} =$ } Why? Because we just did it right now

$\text{Proj}_{\vec{a}}\vec{u} = \frac{3}{14}(-2,3,1)$

② How do you prove orthogonality? ↳ Dot product

$\vec{u}\cdot\vec{a} = \frac{(1,1,2)\cdot(-2,3,1)}{\left(\sqrt{(-2)^2+(3)^2+(1)^2}\right)^2} = \left(\frac{3}{14}\right)$

$\frac{\vec{u}\cdot\vec{a}}{\|\vec{a}^2\|}(\vec{a}) = \frac{3}{14}(\vec{a}) = \frac{3}{14}(-2,3,1) = \left(\frac{-3}{7}, \frac{9}{14}, \frac{3}{14}\right)$

$\vec{u} - \text{proj}_{\vec{a}}\vec{u} = (1,1,2) - \left(\frac{-3}{7}, \frac{9}{14}, \frac{3}{14}\right) = \left(\frac{7+3}{7}, \frac{14-9}{14}, \frac{28-3}{14}\right) = \left[\frac{10}{7}, \frac{5}{14}, \frac{25}{4}\right]$

ⓑ (12:00:∞)

$\left(\frac{-3}{7}, \frac{9}{14}, \frac{3}{14}\right)\cdot\left(\frac{10}{7}, \frac{5}{14}, \frac{25}{14}\right) = \frac{-30}{49} + \frac{45}{196} + \frac{75}{196} = \frac{-120+45+75}{196} = \frac{0}{196} = \boxed{0}$

Does it makes sense that their dot product is $\emptyset$. WHY
↳ Scalar multiple, if one is clothed $\neq \emptyset$, the other is $\emptyset$.

TO PROVE ORTHOGONALITY USE DOT PRODUCT

3.4

The Geometry of Linear Systems (12:05:00)

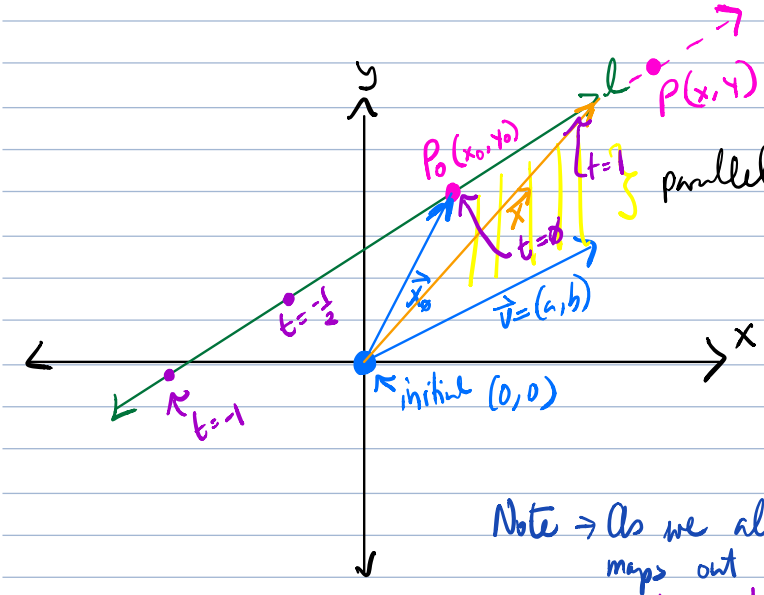
Think about things in higher/lower dimensions.

Idea: Another way of specifying the equations of a line or plane in $\mathbb{R}^n$ is to use a point //What's a normal vector// point and a direction of a vector parallel to the line or plane.

Joseph is changing definitions slightly to avoid ambiguity about points & vectors.

Definition: If $\vec{x}_0$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, and if $\vec{v}$ is nonzero, then the equation $\vec{x} = \vec{x}_0 + t\vec{v}$ defines the line passing through the terminal point of $\vec{x}_0$ and that is parallel to $\vec{v}$.

If $\vec{x}_0 = \vec{\emptyset}$, the line passes through the origin. (t is a constant)



parallel ~ scalar multiples.

Note: $\overrightarrow{P_0P}$ is a scalar multiple of $\vec{v}$. Hence $\overrightarrow{P_0P} = t\vec{v}$

↳ So you allow t to vary the terminal point of the x vector Looks out the coordinates of l

initial point of $\vec{x}_0$ is origin

Note ⇒ As we allow t to vary the terminal point of $\vec{x}$ maps out the line l in $\mathbb{R}^n$. That is why we cannot restrict t.

Let's see what the equation actually looks like.

① What are the components of $\vec{x}$? What's initial/terminal point?

$$\vec{x} = \vec{x}_0 + t\vec{v} \rightarrow (x, y) = (x_0, y_0) + t(a, b)$$

Parametric version:

$$\left.\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \end{aligned}\right\} \text{ Parametric equations for a line in } \mathbb{R}^2$$

Vector form for equation of a line in $\mathbb{R}^2$

Ex Not in book

Find a vector equation AND a parametric equation for the line in $\mathbb{R}^2$ containing the point $(2,-1)$ and that is parallel to the vector $-4\hat{i}-2\hat{j}$

(b) Find two other points on the line.

• Week 10 Class 2 4/4/19

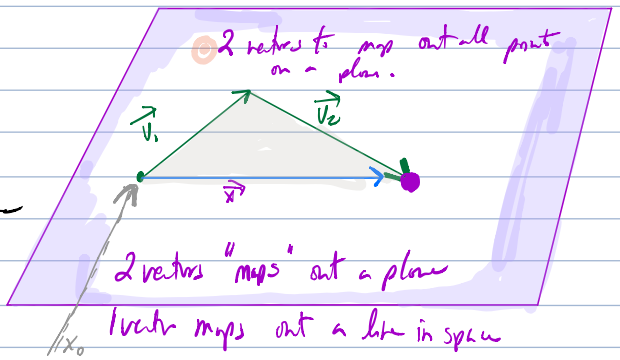
• Exam 2 → Ch. 2 + 3 • Study guide online.
• HWs online.

• Point & vector parallel → to write equation of a line.
• Today: Plane

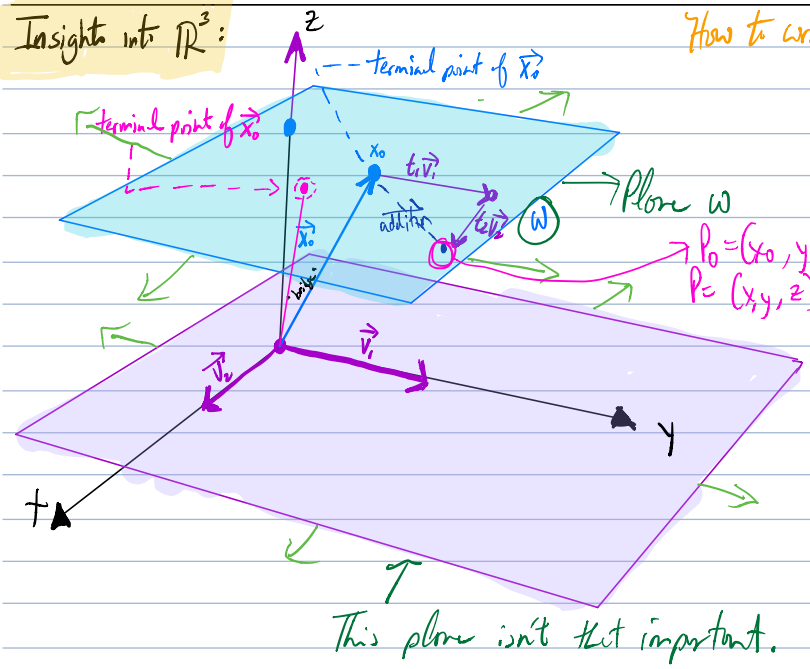
3.4 Continued

Definition → If $\vec{x}_0$, $\vec{v}_1$, $\vec{v}_2$ are all vectors in $\mathbb{R}^n$ (non zero vectors) and if $\vec{v}_1$ and $\vec{v}_2$ are not colinear, then the equation $\vec{x} = \vec{x}_0 + t_1\vec{v}_1 + t_2\vec{v}_2$ defines the plane through the terminal point of $\vec{x}_0$ that is parallel to the plane mapped out by $\vec{v}_1$ and $\vec{v}_2$.

If the $\vec{x}_0 = \vec{0}$, then the plane passes through the origin.



Insights into $\mathbb{R}^3$:



This plane isn't that important.

How to write equations on a plane? Equation of a plane.
• vector equation of a plane.

- $\vec{v}_1 = (a,b,c)$
- $\vec{v}_2 = (d,e,f)$

⊙ plane goes on forever

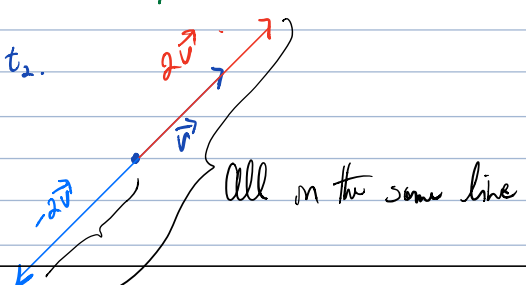
⊙ Terminal point of $\vec{x}_0$ will be given to you.

⊙ "Hyper cube" — a higher dimensional object

⊙ Plane is 2D

* Remember you need two nonlinear vectors to map out a plane. (13:34:00)

⊙ We are varying the parameters t_1, t_2.
⊙ t. are free variables.
⊙ 2 non colinear lines
⊙ Vector addition



// Take the equation, fill in components:

→ $\vec{x} = \vec{x_0} + t_1\vec{v_1} + t_2\vec{v_2}$ ⟶ $(x, y, z) = (x_0, y_0, z_0) + t_1(a, b, c) + t_2(d, e, f)$

↳ vector equation for a plane in $\mathbb{R}^n$.

$$\begin{bmatrix} x = x_0 + at_1 + dt_2 \\ y = y_0 + bt_1 + et_2 \\ z = z_0 + ct_1 + ft_2 \end{bmatrix}$$ → Parametric equations of a plane in $\mathbb{R}^3$

[// From now on, you have to ask and consider about what plane/space is the equation being referred to.]

(Ex) Find vector and parametric equations of the plane that contains the point $(0, 6, 2)$ and is parallel to the plane containing the vectors $\vec{v_1} = (0, 9, -1)$ and $\vec{v_2} = (0, -3, 0)$.

→ $(x, y, z) = (0, 6, 2) + t_1(0, 9, -1) + t_2(0, -3, 0)$

How do we write parametric equation?

$x = 0$
$y = 6 + 9t_1 - 3t_2$
$z = 2 - t_1$

} This is the parametric equation

⊙ All of the coordinates of x will be 0, so the plane is in the yz plane

---

- Both vectors have 0-components, so for this example the planes are the same.

Next & Last Topic of 3.4 → Dot Product Form of a Linear System.

- Consider the linear system

$$\begin{matrix} \vec{r_1} \\ \vec{r_2} \\ \vec{r_3} \end{matrix} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 1 & 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

What type of system is this?

→ Homogeneous Linear System.

↑ Q: How do you get a 0?

→ $(1)(x_1) + 0(x_2) - 1(x_3) + 0(x_4) = 0$

Q: Does this look like something?

→ Dot product. Can't dot matrices, what are you dotting?

$(1)(x_1) + 0(x_2) - 1(x_3) + 0(x_4) = 0$ | $\vec{r_1} \cdot \vec{x} = 0$
$0(x_1) + 1(x_2) + 2(x_3) - 1(x_4) = 0$ | $\vec{r_2} \cdot \vec{x} = 0$
$1(x_1) + 2(x_2) + 3(x_3) - 1(x_4) = 0$ | $\vec{r_3} \cdot \vec{x} = 0$

* dot product form of a linear system

→ $0(x_1) + 1(x_2) + 2(x_3) - 1(x_4) = 0$
→ $1(x_1) + 2(x_2) + 3(x_3) - 1(x_4) = 0$
→ We're dotting the solutions vector

(14:06:00)

The rows in a homogeneous linear system is ORTHOGONAL

* Since the dot products of the rows of A with $\vec{x}$ equals zero, the rows are orthogonal to the solution space.

Theorem: If A is an $m \times n$ matrix then the solution set of the homogeneous linear system $A\hat{x} = \vec{0}$ consists of all vectors in $\mathbb{R}^n$ that are orthogonal to every row vector of A.
(domain)

The relationship between Homogeneous $A\hat{x} = \vec{0}$ and Non Homogeneous $A\hat{x} = \vec{b}$

Consider the linear system:

$$\left[\begin{array}{cccc|c|c} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 2 & -1 & 0 & 3 \\ 1 & 2 & 3 & -1 & 0 & 7 \end{array}\right]$$
(homo | nonhomo)

- dual augmented

Calculator Steps
① Augment 4 rows − 3 × 5, augment with $\vec{0}$.
→ input into the calculator
* Do Not USE AUGMENT COMMAND *
→ Just enter the matrix
→ 3 × 5, 5th column is the augmented $\vec{0}$.
- 2nd matrix, math → RREF →
① Don't use Alpha A → use the matrix.

- RREF of A is RREF of A is RREF of A
↳ RREF is unique

① Find the RREF of the system with homogeneous system → RREF

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$
→ Interpret this?

- $x_1 = x_3$
- $x_2 = -2x_3$
- $x_3 = x_3$
- $x_4 = 0$

Put this in a vector in terms of free vectors →

vector for solution set
$(x_3, -2x_3, x_3, 0)$

factor out the free variable

$x_3(1, -2, 1, 0)$ ← Parameterized equation of the system

we usually see x_3 as t

⑥ Now augment with non homogeneous inputs $\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$

$$\left[\begin{array}{cccc|c|c} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 2 & -1 & 0 & 3 \\ 1 & 2 & 3 & -1 & 0 & 7 \end{array}\right]$$

(homo | nonhomo)

- We already know the RREF of this side

Q: What relationship do you see between homogeneous, nonhomogeneous?

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{interpret}}$$

Strip out the parameters

$x_1 = 1 + x_3$
$x_2 = 3 - 2x_3$
$x_3 = x_3$
$x_4 = \phi$

$(1+x_3, 3-2x_3, x_3, \phi)$

$(1, 3, \phi, \phi) + x_3(1, -2, 1, \phi)$

Constants | Variable.

4/9/2019 (Week 11 Class 1) — Test Thursday — (§3.3, §3.4)

3.4 Continued

⑥

$$\left[\begin{array}{cccc|c|c} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 2 & -1 & 0 & 3 \\ 1 & 2 & 3 & -1 & 0 & 7 \end{array}\right]$$

Homogeneous solution: $x_3(1, -2, 1, 0)$

(x_3 is a parameter // it can be anything)

Nonhomogeneous solution: $x_3(1, -2, 1, \phi) + (1, 3, 0, 0)$

- Homogeneous solution + specific

Note: Theorem: The general solution of a consistent linear system $A\vec{x} = \vec{b}$ (nonhomogeneous system) can be obtained by adding the general solution of $A\vec{x} = 0$ to any specific solution to $A\vec{x} = \vec{b}$.

[A parameter is free to be anything it wants.]

- You need 2 vectors for an equation of a plane.

$\vec{x} = \vec{x}_\phi + t\vec{v}$

$x_3(1, -2, 1, \phi)$ // This is an equation of a line in $\mathbb{R}_4$

$t\vec{v}$

terminal point

$$\left[\begin{array}{c} x_3(1, -2, 1, 0) + (1, 3, 0, 0) \\ t(\vec{v}) \quad + \quad x_\phi \end{array}\right]$$